

**Sirindhorn International Institute of Technology**  
**Thammasat University**  
 School of Information, Computer and Communication Technology

ECS332 2016/1      Part III.1      Dr. Prapun

Transition from analog to digital

## 6 Sampling and Reconstruction

### 6.1 Sampling

**Definition 6.1. Sampling** is the process of taking a (sufficient) number of discrete values of points on a waveform that will define the shape of wave form.

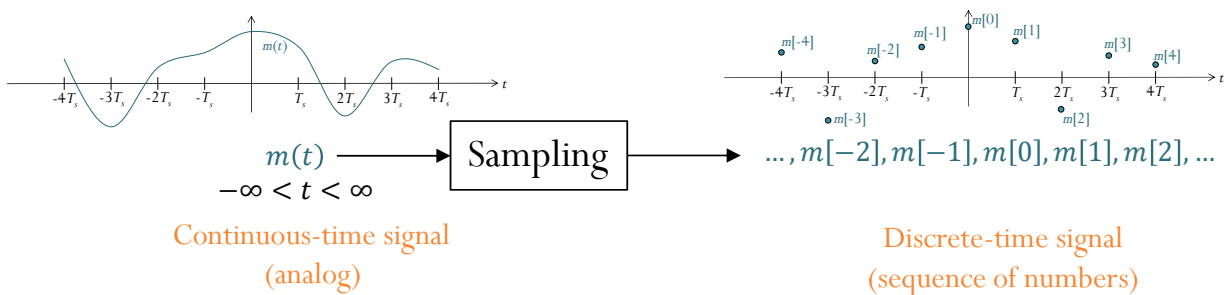


Figure 40: The Sampling Process

- In this class, the signal is sampled at a uniform rate, once every  $T_s$  seconds.

$$m[n] = m(nT_s) = m(t)|_{t=nT_s}$$

- We refer to  $T_s$  as the sampling period, and to its reciprocal  $f_s = 1/T_s$  as the sampling rate which is measured in samples/sec [Sa/s].
- At this stage, we assume “infinite” precision (no quantization) for each value of  $m[n]$ .

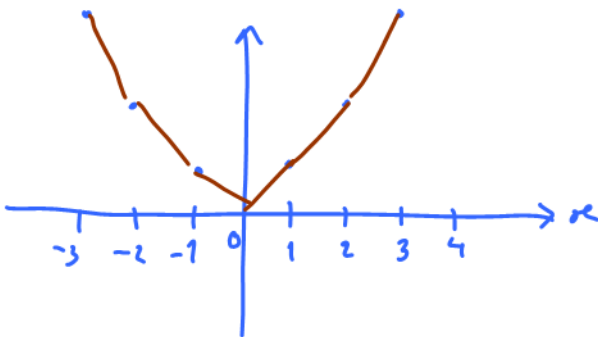
- The reverse process is called “reconstruction”.

**6.2. Sampling = loss of information?** If not, how can we recover the original waveform back.

- The more samples you take, the more accurately you can define a waveform.
- Obviously, if the sampling rate is too low, you may experience distortion (aliasing).
- The <sup>(version 1)</sup> sampling theorem, to be discussed in the section, says that when the waveform is band-limited, if the sampling rate is fast enough, we can reconstruct the waveform back and hence there is no loss of information.
  - This allows us to replace a continuous time signal by a discrete sequence of numbers.
  - Processing a continuous time signal is therefore equivalent to processing a discrete sequence of numbers.
  - In the field of communication, the transmission of a continuous time message reduces to the transmission of a sequence of numbers.

**Example 6.3.** Mathematical functions are frequently displayed as continuous curves, even though a finite number of discrete points was used to construct the graphs. If these points, or samples, have sufficiently close spacing, a smooth curve drawn through them allows us to interpolate intermediate values to any reasonable degree of accuracy. It can therefore be said that the continuous curve is adequately described by the sample points alone.

**Example 6.4.** Plot  $y = x^2$ .



Step ① is sampling

① “Evaluate”  $y = x^2$  at many values of  $x$

② “connect” the dots

Step ② is interpolation reconstruction

**Example 6.5.** In Figure 41, we plot the function  $g(t) = \sin(100\pi t)$  from 0 to 1 by connecting the values of the function at fifty uniformly-spaced points.

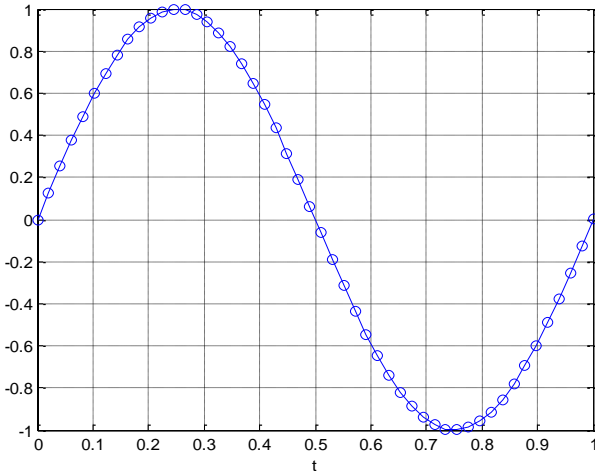


Figure 41: Plot of the function  $g(t) = \sin(100\pi t)$  from 0 to 1 by connecting the values at fifty uniformly-spaced points.

Although the plot shows the correct shape of the sine wave, the perceived frequency is just 1 Hz.

(Version 2)

**Theorem 6.6. Sampling Theorem:** In order to (correctly and completely) represent an analog signal, the sampling frequency,  $f_s$ , must be at least twice the highest frequency component of the analog signal.

**6.7.** If the conditions of the sampling theorem are not satisfied, we experience an effect called **aliasing** in which different signals become indistinguishable (or aliases of one another) when sampled.

- The term “aliasing” also refers to the distortion or artifact that results when the signal reconstructed from samples is different from the original continuous signal.

**Example 6.8.** In Example 6.5, the frequency of the sine wave is 50 Hz. Therefore, we need the sampling frequency to be at least 100.

However, there, the number of samples per sec. is 49 ← too small to avoid aliasing.

**6.9.** For now, instead of trying to infer the “perceived” frequency by analyzing the plot of the function in the time domain, it is easier to use our `plotspect` function to visualize the location of the peaks (of the delta functions) in the frequency domain.

**Example 6.10.** Suppose the sampling frequency is 200 samples/sec. The analog signal should not have the frequency higher than 100 Hz. This is illustrated in Figure 42 in which cosine functions of different frequencies are sampled with  $f_s = 200$ .

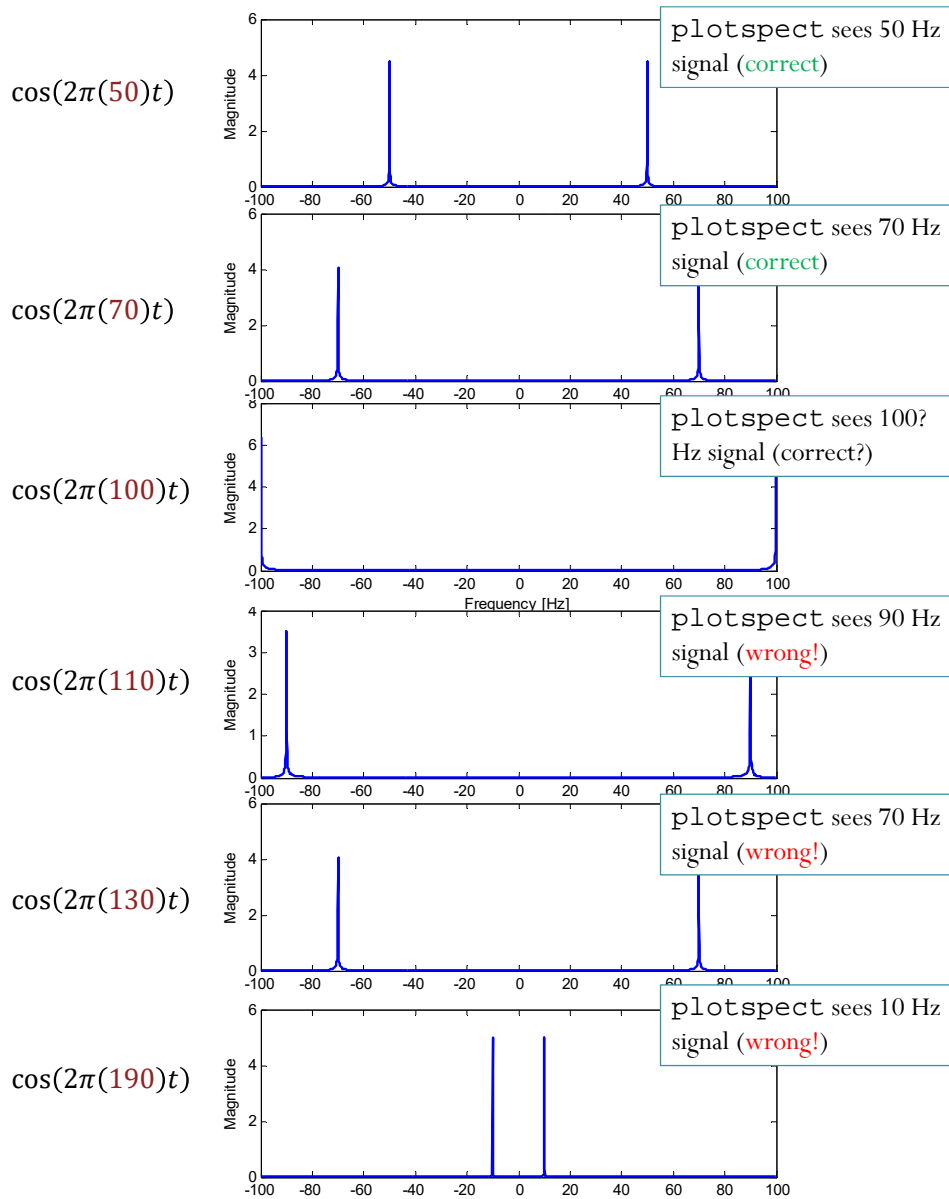


Figure 42: Our `plotspect` function can be used to visualize the “perceived” frequency of a sampled signal.

**6.11.** Steps to find the “perceived” frequency of the sampled signal when the sampling rate is  $f_s$ :

Because  $\cos(-x) = \cos(x)$ ,

w/o loss of generality, we may assume that  $f_0 \geq 0$

(a) For  $\cos(2\pi(f_0)t)$ , we may use the "folding technique":

and that  
the perceived freq.  
is always  $\geq 0$ .

- (i) Consider the window of frequency from 0 to  $\frac{f_s}{2}$ .
- (ii) Start from 0, increase the frequency to  $f_0$ .  
Fold back at 0 and  $\frac{f_s}{2}$  if necessary.

Remark: By the symmetry in the spectrum of cosine, we can always give a nonnegative answer for the perceived frequency.

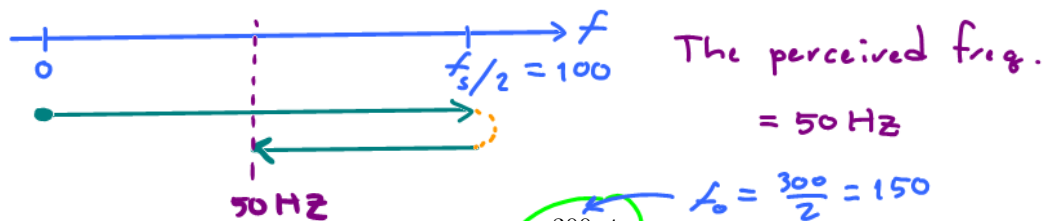
(b) For  $e^{j2\pi(f_0)t}$  we use the "tunneling technique":

- (i) Consider the window of frequency from  $-\frac{f_s}{2}$  to  $\frac{f_s}{2}$ .
- (ii) Start from 0.
  - i. If  $f_0 > 0$ , increase the frequency to  $f_0$  (going to the right).  
Restart at  $-\frac{f_s}{2}$  when  $\frac{f_s}{2}$  is reached.
  - ii. If  $f_0 < 0$ , decrease the frequency to  $f_0$  (going to the left).  
Restart at  $+\frac{f_s}{2}$  when  $-\frac{f_s}{2}$  is reached.

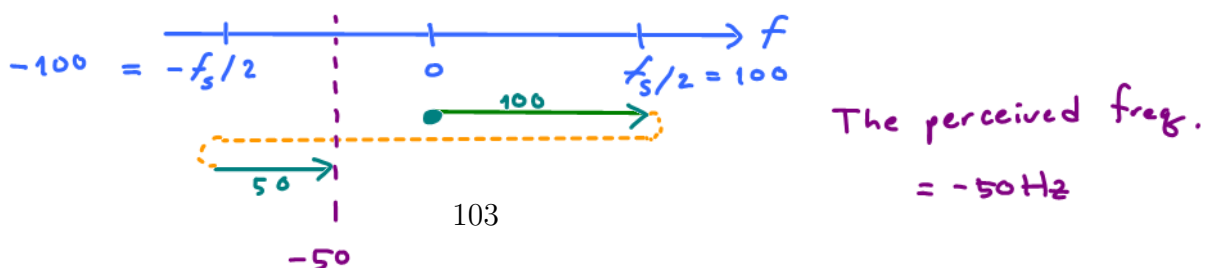
Remark: A cosine function at frequency  $f_0$  can also be thought of as a combination of two complex exponential at frequency  $f_0$  and  $-f_0$ . Therefore, we can also use the tunneling technique to analyze the cosine function as well by looking at its individual complex-exponential components.

(c) We will study a more general analysis in Section 6.3.

**Example 6.12.** Find the perceived frequency of  $\cos(300\pi t)$  when the sampling rate is 200 [Sa/s].  $f_s = 200$

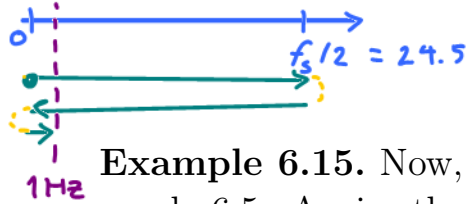


**Example 6.13.** Find the perceived frequency of  $e^{300\pi t}$  when the sampling rate is 200 [Sa/s].

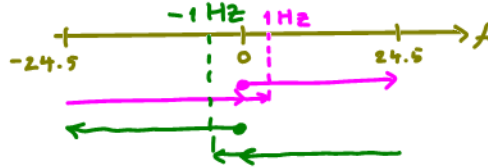


**Example 6.14.** Let's consider a signal that is closer to Example 6.5. Suppose we consider  $\cos(100\pi t)$ . The sampling rate used is 49 [Sa/s]. Find the perceived frequency.

① Folding technique:



② Tunneling Technique



$$\cos(2\pi(50)t) = \frac{e^{j2\pi(50)t} + e^{j2\pi(-50)t}}{2}$$

$$\cos(2\pi(1)t) = \frac{e^{j2\pi(1)t} + e^{j2\pi(-1)t}}{2}$$

**Example 6.15.** Now, let's consider the signal  $\sin(100\pi t)$  discussed in Example 6.5. Again, the sampling rate used is 49 [Sa/s]. Find the perceived signal.

$$\sin(100\pi t) = \sin(2\pi(50)t) = \frac{e^{j2\pi(50)t} - e^{j2\pi(-50)t}}{2j}$$

perceived as

$$\sin(2\pi(1)t) = \frac{e^{j2\pi(1)t} - e^{j2\pi(-1)t}}{2j}$$

**Example 6.16.** Application of the sampling theorem: In telephony, the usable voice frequency band ranges from approximately 300 Hz to 3400 Hz. The bandwidth allocated for a single voice-frequency transmission channel is usually 4 kHz, including guard bands, allowing a sampling rate of 8 kHz to be used as the basis of the pulse code modulation system used for the digital PSTN.

**Definition 6.17.**

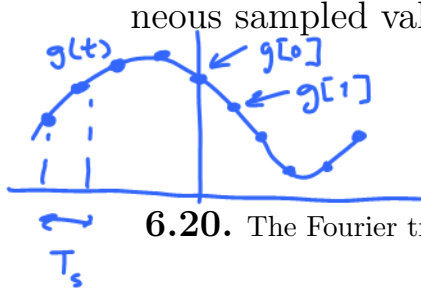
- Given a sampling frequency,  $f_s$ , the **Nyquist frequency** is  $f_s/2$ .
- Given the highest (positive-)frequency component  $f_{\max}^B$  of an analog signal,
  - the **Nyquist sampling rate** is  $2f_{\max}$  and
  - the **Nyquist sampling interval** is  $1/(2f_{\max})$ .

**6.18.** For the remaining analysis in this section, we will use  $g(t)$  to denote the signal under consideration. You may replace  $g(t)$  below by  $m(t)$  if you want to think of it as an analog message to be transmitted by a communication system. We use  $g(t)$  here because the results provided here work in broader setting as well.

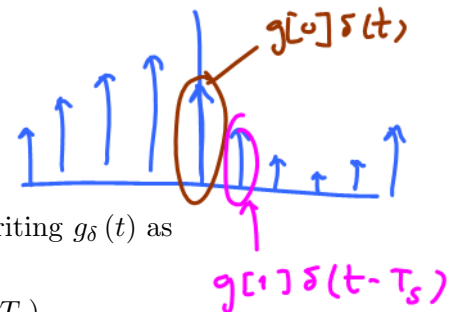
*Formal analysis for other signals*

## 6.2 Ideal Sampling

**Definition 6.19.** In **ideal sampling**, the (ideal instantaneous) sampled signal is represented by a train of impulses whose area equal the instantaneous sampled values of the signal



$$g_{\delta}(t) = \sum_{n=-\infty}^{\infty} g[n] \delta(t - nT_s).$$



**6.20.** The Fourier transform  $G_{\delta}(f)$  of  $g_{\delta}(t)$  can be found by first rewriting  $g_{\delta}(t)$  as

$$\begin{aligned} g_{\delta}(t) &= \sum_{n=-\infty}^{\infty} g(nT_s) \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} g(t) \delta(t - nT_s) \\ &= g(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s). \end{aligned}$$

Multiplication in the time domain corresponds to convolution in the frequency domain. Therefore,

$$G_{\delta}(f) = \mathcal{F}\{g_{\delta}(t)\} = G(f) * \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right\}.$$

For the last term, the Fourier transform can be found by applying what we found in Example 4.41<sup>25</sup>:

$$\sum_{n=-\infty}^{\infty} \delta(t - nT_s) \xleftrightarrow[\mathcal{F}^{-1}]{\mathcal{F}} f_s \sum_{k=-\infty}^{\infty} \delta(f - kf_s).$$

This gives

$$G_{\delta}(f) = G(f) * f_s \sum_{k=-\infty}^{\infty} \delta(f - kf_s) = f_s \sum_{k=-\infty}^{\infty} G(f) * \delta(f - kf_s).$$

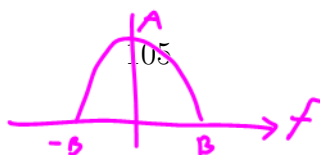
Hence, we conclude that

$$g_{\delta}(t) = \sum_{n=-\infty}^{\infty} g[n] \delta(t - nT_s) \xleftrightarrow[\mathcal{F}^{-1}]{\mathcal{F}} G_{\delta}(f) = f_s \sum_{k=-\infty}^{\infty} G(f - kf_s). \quad (75)$$

In words,  $G_{\delta}(f)$  is simply a sum of the scaled and shifted replicas of  $G(f)$ .

<sup>25</sup>We also considered an easy-to-remember pair and discuss how to extend it to the general case in 4.42.

*suppose  $g(t) \xrightarrow{\mathcal{F}} G(f)$*



modulation  $y(t) = g(t) \cos(2\pi f_c t)$   
 $Y(f) = \frac{1}{2} G(f-f_c) + \frac{1}{2} G(f+f_c)$

**6.21.** As usual, we will assume that the signal  $g(t)$  is band-limited to  $B$  Hz ( $G(f) = 0$  for  $|f| > B$ ).

(a) When  $B < f_s/2$  as shown in Figure 43, the replicas do not overlap and hence we do not need to spend extra effort to find their sum.

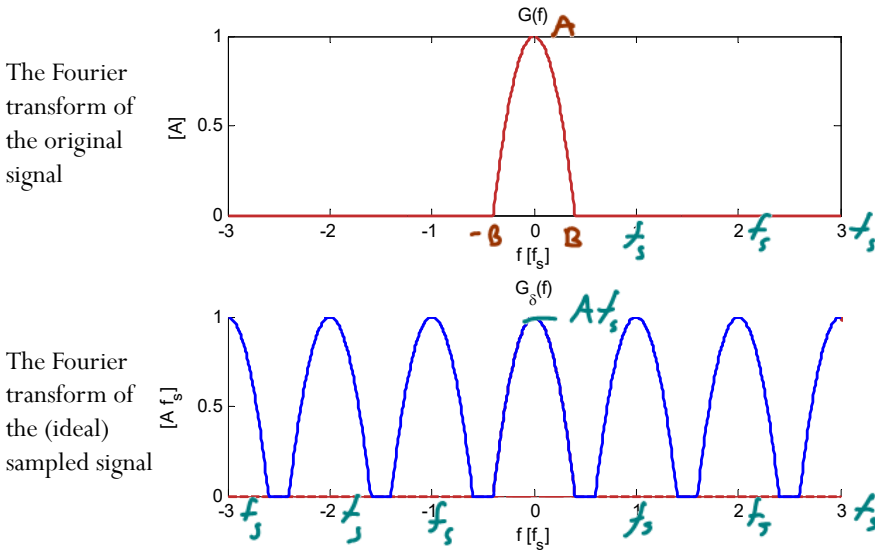


Figure 43: The Fourier transform  $G_\delta(f)$  of  $g_\delta(t)$  when  $B < f_s/2$

(b) When  $B > f_s/2$  as shown in Figure 44, overlapping happens in the frequency domain. This spectral overlapping of the signal is (also) commonly referred to as “aliasing” mentioned in 6.7. To find  $G_\delta(f)$ , don't forget to add the replicas

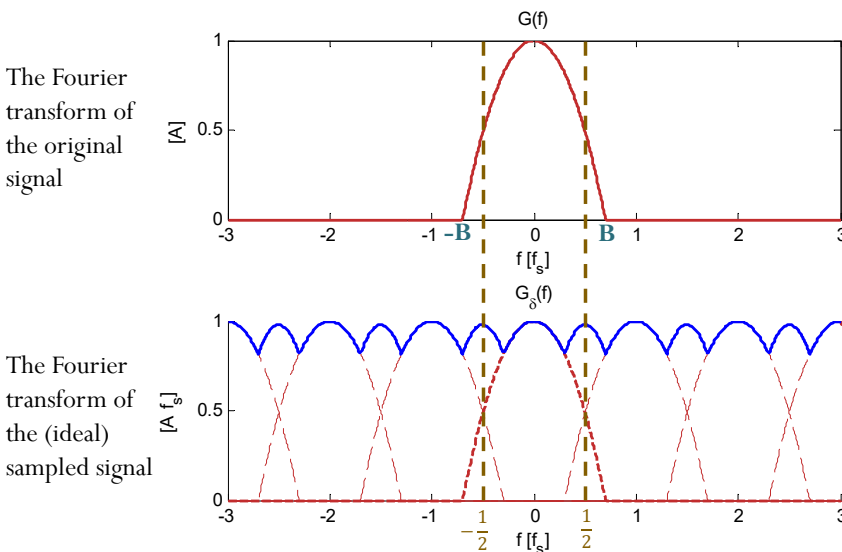


Figure 44: The Fourier transform  $G_\delta(f)$  of  $g_\delta(t)$  when  $B > f_s/2$



**6.22. Remarks:**

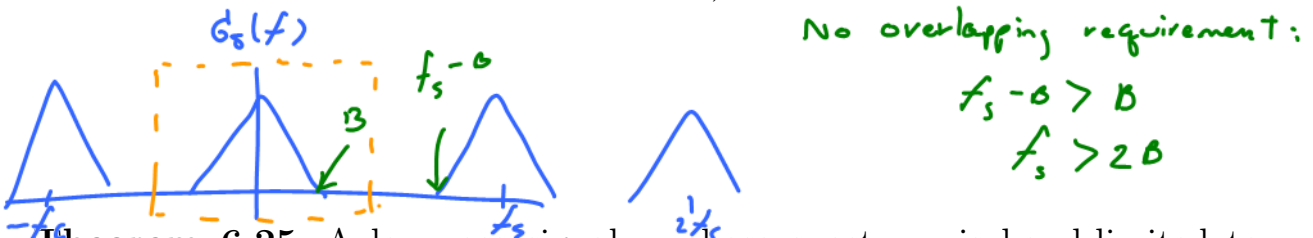
- (a)  $G_\delta(f)$  is “periodic” (in the frequency domain) with “period”  $f_s$ .
  - So, it is sufficient to look at  $G_\delta(f)$  between  $\pm \frac{f_s}{2}$
- (b) The MATLAB script `plotspect` that we have been using to visualize magnitude spectrum also relies on sampled signal. Its frequency domain plot is between  $\pm \frac{f_s}{2}$ .
- (c) Although this sampling technique is “ideal” because it involves the use of the  $\delta$ -function. We can extract many useful conclusions.
- (d) One can also study the discrete-time Fourier transform (DTFT) to look at the frequency representation of the sampled signal.

**6.3 Reconstruction**



**6.23.** From (75), we see that when the sampling frequency  $f_s$  is large enough, the replicas of  $G(f)$  will not overlap in the frequency domain. In such case, the original  $G(f)$  is still intact and we can use a low-pass filter with gain  $T_s$  to recover  $g(t)$  back from  $g_\delta(t)$ .

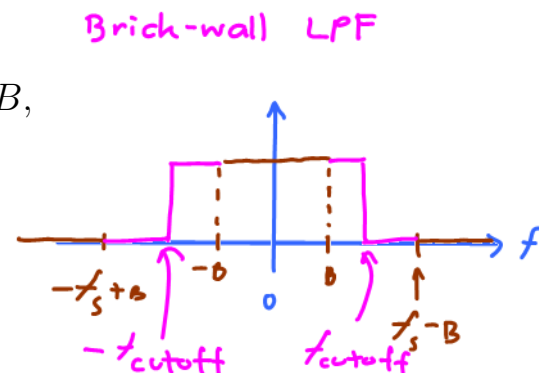
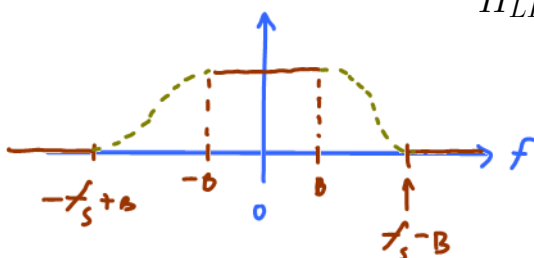
**6.24.** To prevent aliasing (the corruption of the original signal because its replicas overlaps in the frequency domain), we need



**Theorem 6.25.** A low-pass signal  $g$  whose spectrum is band-limited to  $B$  Hz ( $G(f) = 0$  for  $|f| > B$ ) can be reconstructed (interpolated) exactly (without any error) from its sample taken uniformly at a rate (sampling frequency/rate)  $f_s > 2B$  Hz (samples per second). [5, p 302]

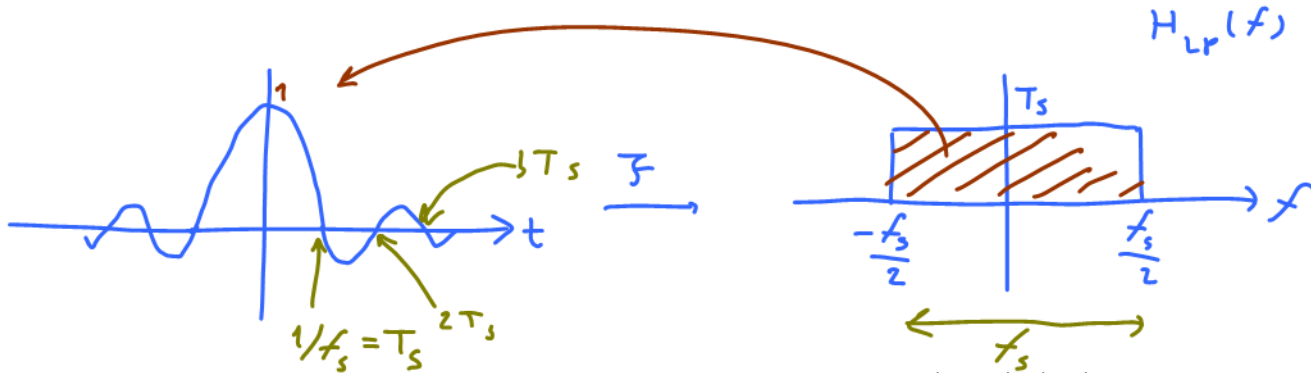
**6.26. Ideal Reconstruction:** Continue from 6.23. Assuming that  $f_s > 2B$ , the low-pass filter that we should use to extract  $g(t)$  from  $G_\delta(t)$  should be

$$H_{LP}(f) = \begin{cases} T_s, & |f| \leq B, \\ \text{any}, & B < |f| < f_s - B, \\ 0, & |f| \geq f_s - B, \end{cases}$$



In particular, for “brick-wall” LPF, the cutoff frequency  $f_{\text{cutoff}}$  should be between  $B$  and  $f_s - B$ .

**6.27. Reconstruction Equation:** Suppose we use  $\frac{f_s}{2}$  as the cutoff frequency for our “brick-wall” LPF in 6.26,



The impulse response of the LPF is  $h_{LP}(t) = \text{sinc}\left(2\pi\left(\frac{f_s}{2}\right)t\right) = \text{sinc}(\pi f_s t)$ .  
The output of the LPF is

$$\begin{aligned} \hat{g}_\bullet(t) &= g_\delta(t) * h_{LP}(t) = \left( \sum_{n=-\infty}^{\infty} g[n] \delta(t - nT_s) \right) * h_{LP}(t) \\ &= \sum_{n=-\infty}^{\infty} g[n] h_{LP}(t - nT_s) = \sum_{n=-\infty}^{\infty} g[n] \text{sinc}(\pi f_s(t - nT_s)). \end{aligned}$$

When  $f_s > 2B$ , this output will be exactly the same as  $g(t)$ :

reconstruction eq.  $g(t) = \sum_{n=-\infty}^{\infty} g[n] \text{sinc}(\pi f_s(t - nT_s))$  (76)

- This formula allows perfect reconstruction of the original continuous-time function from the samples.
- At the sampling instants  $t = nT_s$ , all sinc functions are zero at these times save one, and that one yields  $g(nT_s)$  which is the correct values.
- Note that at time  $t$  between the sampling instants,  $g(t)$  is interpolated by summing the contributions from all the sinc functions.
- The LPF is often called an interpolation filter, and its impulse response is called the interpolation function.

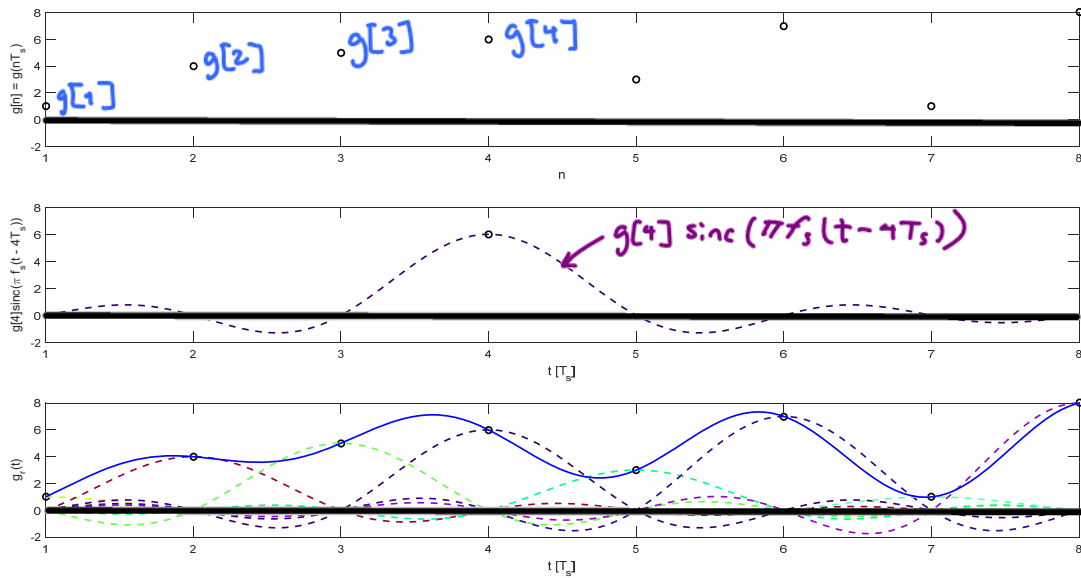


Figure 45: Application of the reconstruction equation

**Example 6.28.** In Figure 45, a signal  $g_r(t)$  is reconstructed from the samples value  $g[n]$  via the reconstruction equation (76).

**Example 6.29.** We now return to the sampling of the cosine function (sinusoid).

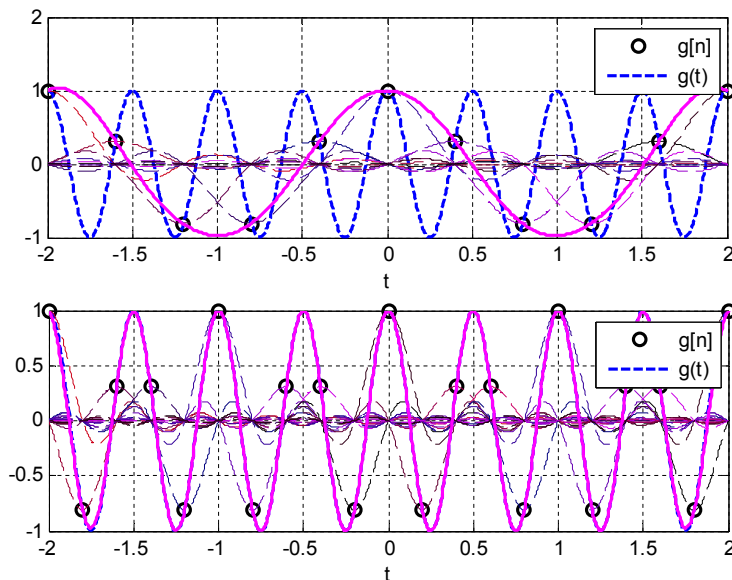


Figure 46: Reconstruction of the signal  $g(t) = \cos(2\pi(2)t)$  by its samples  $g[n]$ . The upper plot uses  $T_s = 0.4$ . The lower plot uses  $T_s = 0.2$ .

**Theorem 6.30. Sampling theorem for uniform periodic sampling:** If a signal  $g(t)$  contains no frequency components for  $|f| \geq B$ , it is completely

described by instantaneous sample values uniformly spaced in time with sampling period  $T_s \leq \frac{1}{2B}$ . In which case,  $g(t)$  can be exactly reconstructed from its samples  $(\dots, g[-2], g[-1], g[0], g[1], g[2], \dots)$  by the reconstruction equation (76).

**6.31. Remarks:**

- Need a lot of  $g[n]$  for the reconstruction.

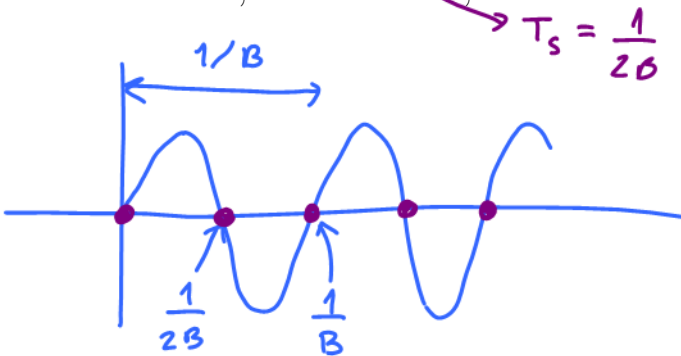
- Practical signals are time-limited.  $\Rightarrow$  not band-limited  $\Rightarrow f_{max} = \infty \Rightarrow f_s$  must also be  $\infty$ 
  - Filter the message as much as possible before sampling.

(B)  
 ①  $f_s$  must also be  $\infty$   
 ↓  
 ② use anti-aliasing filter to eliminate the high freq. content of  $G(f)$  first before sampling.

**6.32. The possibility of  $f_s = 2B$ :**

- If the spectrum  $G(f)$  has no impulse (or its derivatives) at the highest frequency  $B$ , then the overlap is still zero as long as the sampling rate is greater than or equal to the Nyquist rate, that is,  $f_s \geq 2B$ .
- If  $G(f)$  contains an impulse at the highest frequency  $\pm B$ , then  $f_s = 2B$  would cause overlap. In such case, the sampling rate  $f_s$  must be greater than  $2B$  Hz.

**Example 6.33.** Consider a sinusoid  $g(t) = \sin(2\pi(B)t)$ . This signal is bandlimited to  $B$  Hz, but all its samples are zero when uniformly taken at a rate  $f_s = 2B$ , and  $g(t)$  cannot be recovered from its (Nyquist) samples. Thus, for sinusoids, the condition of  $f_s > 2B$  must be satisfied.



$g[n] = 0 \quad \forall n$  "for all"  
 ↓  
 $g_s(t) = \sum_n g[n] \delta(t - nT_s) \equiv 0 \quad \forall t$   
 ↓  
 $G_s(f) \equiv 0 \quad \forall f \rightarrow \hat{G}(f) \equiv 0 \quad \forall f$   
 ↓  
 $\hat{g}(t) \equiv 0 \quad \forall t$

Let's check with our formula (75) for  $G_s(f)$ . First, recall that

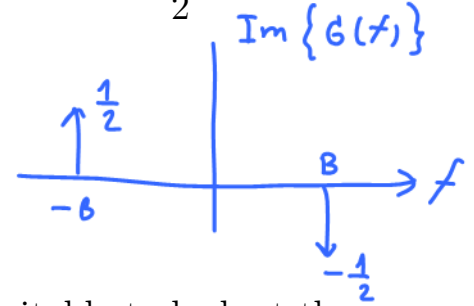
$$\sin x = \frac{e^{jx} - e^{-jx}}{2j} = \frac{1}{2j}e^{jx} - \frac{1}{2j}e^{-jx}.$$

Therefore,

$$g(t) = \sin(2\pi(B)t) = \frac{1}{2j}e^{j2\pi(B)t} - \frac{1}{2j}e^{-j2\pi(B)t} = -\frac{1}{2}je^{j2\pi(B)t} + \frac{1}{2}je^{j2\pi(-B)t}$$

and

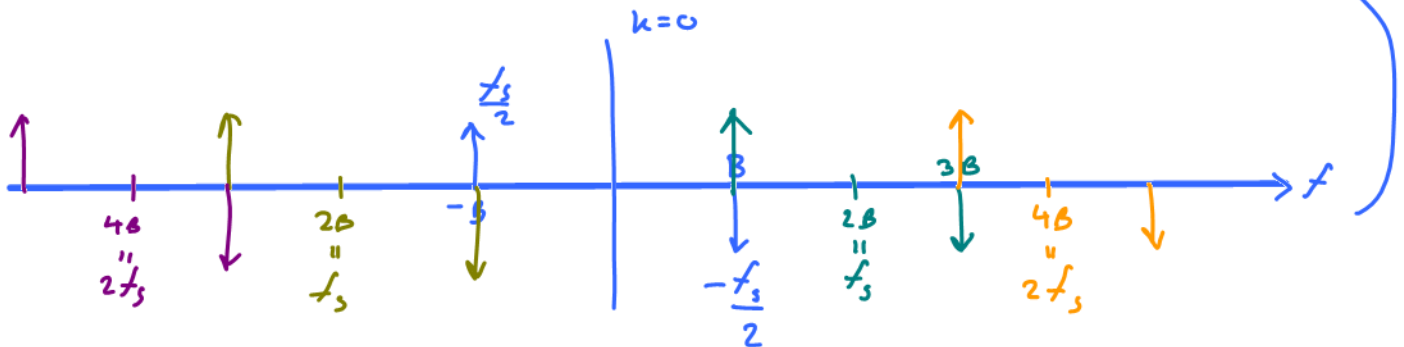
$$G(f) = \left(-\frac{1}{2}j\right)\delta(f-B) + \left(\frac{1}{2}j\right)\delta(f-(-B))$$



Note that  $G(f)$  is pure imaginary. So, it is more suitable to look at the plot of its imaginary part. (We do not look at its magnitude plot because the information about the sign is lost. We also do not consider the real part because we know that it is 0.)

$$G_s(f) = \sum_k f_s G(f - kf_s)$$

$$\begin{aligned} \text{Re}\{G_s(f)\} &= \sum_k f_s \text{Re}\{G(f - kf_s)\} = 0 \\ \text{Im}\{G_s(f)\} &= \sum_k f_s \text{Im}\{G(f - kf_s)\} = 0 \end{aligned}$$



$$G_s(f) = \text{Re}\{G_s(f)\} + j\text{Im}\{G_s(f)\} = 0 + j0 = 0.$$

**6.34.** A maximum of  $2B$  independent pieces (samples/symbols) of information per second can be transmitted, errorfree, over a noiseless channel of bandwidth  $B$  Hz [4, p 260].

- Start with  $2B$  pieces of information per second. Denote the sequence of such information by  $m_n$ .
- Construct a signal  $m(t)$  whose (Nyquist) sample values  $m[n] = m\left(n\frac{1}{2B}\right)$  agrees with  $m_n$  by the reconstruction equation (76).

**6.35.** A bandpass signal whose spectrum exists over a frequency band  $f_c - \frac{B}{2} < |f| < f_c + \frac{B}{2}$  has a bandwidth  $B$  Hz. Such a signal is also uniquely determined by samples taken at above the Nyquist frequency  $2B$ . The sampling theorem is generally more complex in such case. It uses two interlaced sampling trains, each at a rate of  $f_s > B$  samples per second (known as second-order sampling). [5, p 304]

## 6.4 Triangular (Linear) Interpolation

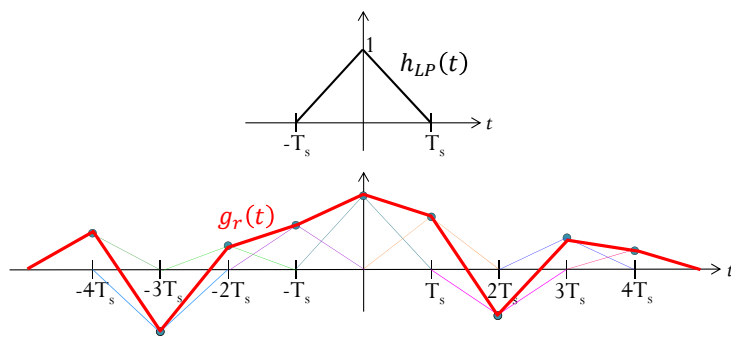


Figure 47: Triangular (Linear) Interpolation

**6.36.** When linear interpolation is used, high frequency content of  $G(f)$  is attenuated and (small part of) the replicas at even higher frequencies (which do not exist before) are also introduced.

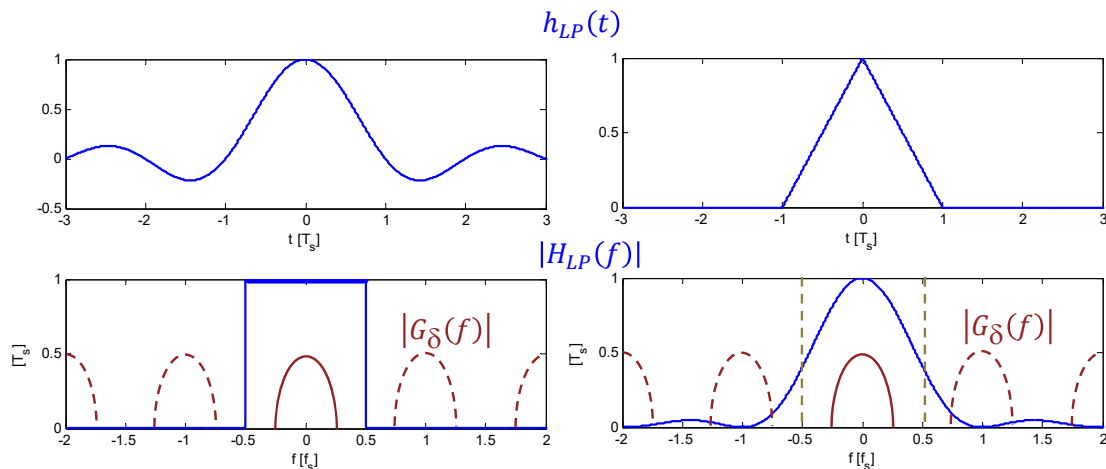


Figure 48: Triangular (Linear) Interpolation: Effects on  $G_r(t)$